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Three-Manifold Invariants from Chern-Simons Field Theory with Arbitrary Semi-Simple Gauge Groups

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Invariants for framed links in S^3 obtained from Chern-Simons gauge field theory based on an arbitrary gauge group (semi-simple) have been used to construct a three-manifold invariant. This is a generalization of a similar construction developed earlier for $SU(2)$ Chern-Simons theory. The procedure exploits a theorem of Lickorish and Wallace and also those of Kirby, Fenn and Rourke which relate three-manifolds to surgeries on framed unoriented links. The invariant is an appropriate linear combination of framed link invariants which does not change under Kirby calculus. This combination does not see the relative orientation of the component knots. The invariant is related to the partition function of Chern-Simons theory. This thus provides an efficient method of evaluating the partition function for these field theories. As some examples, explicit computations of these manifold invariants for a few three-manifolds have been done.

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1 Introduction

In recent times topological quantum theories have proved to be a very powerful tool for the study of geometry and topology of low dimensional manifolds. An example of such theories is the Chern-Simons gauge field theory which provides a general framework for knots and links in three dimensions [1]. Vacuum expectation values of Wilson link operators in this theory yield a class of polynomial link invariants. It was E. Witten who in his pioneering paper about ten years ago developed this framework and also demonstrated that the famous Jones polynomial was related to the expectation value of a Wilson loop operator (in spin 1/2 representation) in an $SU(2)$ Chern-Simons field theory[2]. Since then many attempts have been made to obtain exact and explicit non-perturbative solutions to such field theories [3, 4, 5, 6, 7, 8]. Powerful methods for completely analytical and non-perturbative computations of the expectation values of Wilson link operators have been developed. One such method in its complete manifestation has been presented in ref.[4].

The power of field theoretic framework through Chern-Simons theories is indeed so deep that it allows us to study knots and links not only in simple manifolds such as a three-sphere but also in any arbitrary three-manifold. For example, the link invariants obtained in these field theories can be used to construct three-manifold invariants. One such construction involves an application of Lickorish-Wallace surgery presentation of three-manifolds in terms of unoriented framed links embedded in S^3. Surgery on more than one framed knot or link can yield the same manifold. However, the rules of equivalence of framed links which yield the same three-manifold on surgery are given by theorems of Kirby, Fenn and Rourke and are known as Kirby moves. Thus a three-manifold can be characterized by an appropriate combination of invariants of the associated framed knots and links which is unchanged under Kirby moves. A three-manifold invariant of this type has been recently constructed from invariants for framed links in an $SU(2)$ Chern-Simons theory in a three-sphere [6, 8]. The algebraic formula for the invariant so obtained is rather easy to compute for an arbitrary three-manifold. The construction developed is general enough to yield other three-manifold invariants from the link invariants of Chern-Simons gauge theories based on other semi-simple gauge groups. This extension is what will be presented in the present paper.

Other three-manifold invariants have also been constructed in recent years. For example, exploiting the surgery presentations of three-manifolds in terms of unoriented framed

links, Lickorish had earlier obtained a manifold invariant using bracket polynomials of cables [9]. Evaluation of this invariant involves a tedious calculation through recursion relations. Using representation theory of composite braids [10], it has been possible to directly evaluate the bracket polynomials for cables without going through the recursion relations. This direct calculation has been used to demonstrate the equivalence of the invariant obtained from $SU(2)$ Chern-Simons theory in ref.[6, 8] to the Lickorish's three-manifold invariant up to a variable redefinition [11]. Further Lickorish's invariant is considered to be a reformulation of Reshetikhin-Turaev invariant[12], which in turn is known to be equivalent to the partition function of $SU(2)$ Chern-Simons theory, known as Witten invariant. Thus this establishes[11], by an indirect method, that the field theoretic three-manifold invariant obtained from link invariants in $SU(2)$ Chern-Simons gauge theory using theorems of Lickorish and Wallace, Kirby, Fenn and Rourke is actually partition function of $SU(2)$ Chern-Simons theory, a fact already noticed for many three-manifolds in ref.[6]. This equivalence is up to an over all normalization.

The link invariants in general depend on the framing convention used. The *frame* of a knot is an associated closed curve going along the length of knot and wrapping around it certain number of times. In the field theoretic language, framing has to do with the regularization prescription used to define the coincident loop correlators [8]. In one such framing convention known as *standard framing* the self-linking number of every knot (*i.e.*, linking number of the knot and its framing curve) is zero. This convention was used in obtaining the link invariants in ref. [4]. The invariants so obtained are ambient isotopic invariant, that is, these are unchanged under all the three Reidemeister moves. However, in our present discussion, we are interested in *framed links* which are only regular isotopic, that is, two framed links are equivalent if and only if they are related by two of the Reidmeister moves (excluding the one that changes the writhe). The framing convention for describing such framed links is *vertical framing*. Here the frame is to be just vertically above the strands of every knot projected on to a plane. The link invariants in this framing exhibit only regular isotopy invariance. These framed link invariants are in general sensitive to the relative orientations of component knots of a link. Reversing the orientation in a knot component changes the representation living on the associated Wilson loop operator to its conjugate representation. We construct an appropriate linear combination of these invariants for different group representations on the framed link which is unchanged under Kirby calculus. This combination then characterises the manifold related to the given link by surgery. Though the individual

terms in this combination in general depend on the relative orientations of the knots, the combination does not. This is consistent with Lickorish-Wallace theorem for surgery presentation of three-manifolds which involves only *unoriented* links.

The plan of the paper is as follows: In the next section, we shall briefly discuss Chern-Simons theory based on any arbitrary semi-simple gauge group. Methods of computing the expectation value of Wilson loop operators for framed knots and links will be outlined. These are generalizations of the methods for $SU(2)$ Chern-Simons theory presented in ref. [4, 6]. In Sec.3, we shall present a theorem of Lickorish and Wallace and Kirby calculus which are the necessary ingredients in the construction of three-manifolds by surgery. Using the field theoretic framed link invariants, we derive an algebraic formula for a three-manifold invariant. Sec. 4 will contain some concluding remarks.

2 Chern-Simons field theory and link invariants

In this section, we shall present some of the salient features of Chern-Simons theory on S^3 based on arbitrary semi-simple gauge group $\mathcal{G}$ and the invariants of *framed links* embedded in S^3. These framed link invariants will be used in the construction of three-manifold invariants in the next section.

For a matrix valued connection one-form A of the gauge group $\mathcal{G}$, the Chern-Simons action S on S^3 is given by

$$kS \ = \ \frac{k}{4\pi}\int_{S^3} tr(AdA \ + \ \frac{2}{3}A^3) \ . \qquad (1)$$

The coupling constant k takes integer values. Clearly action (1) does not have any metric of S^3 in it. The topological operators are the metric independent Wilson loop (knot) operators defined as

$$W_R[C] \ = \ tr_R Pexp \oint_C A_R \qquad (2)$$

for a knot C carrying representation R; A_R is the connection field in representation R of the group. Reversing the orientation of a knot corresponds to placing conjugate representation R^* on it.

For a link L made up of component knots $C_1, C_2, \ldots C_r$ carrying $R_1, R_2, \ldots R_r$ representations respectively, we have the Wilson link operator defined as

$$W_{R_1 R_2 \ldots R_r}[L] \ = \ \prod_{\ell=1}^{r} \ W_{R_\ell}[C_\ell] \ . \qquad (3)$$

We are interested in the functional averages of these link operators:

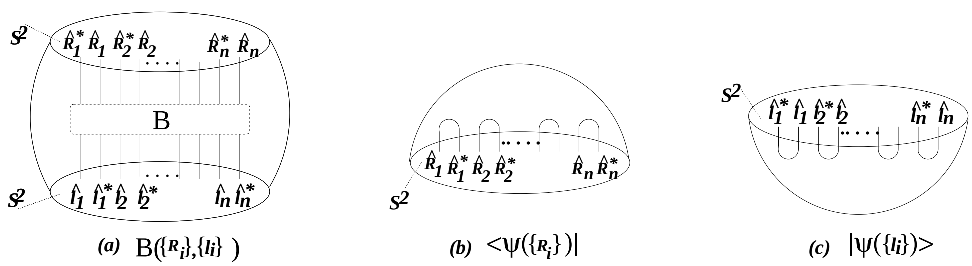
$$V[L;\mathbf{f};R_1,R_2\ldots R_r] \equiv V_{R_1R_2\ldots R_r}[D_L] \quad = \quad Z^{-1}\int_{S^3}[dA]W_{R_1R_2\ldots R_r}[L]e^{ikS}, \tag{4}$$
$$\text{where } \; Z \; = \; \int_{S^3}[dA]e^{ikS} \; ,$$

and D_L denotes link diagram corresponding to framed link $[L,f]$. Link diagrams are obtained from a regular projection of a link in a plane with transverse double points (refered to as crossings) as the only self-intersections. We work in the vertical frame where the frame curve is vertically above the plane of the diagram. Hence, in vertical framing, $\mathbf{f} = (f_1, f_2, \ldots, f_r)$ on link L is a set of integers denoting the sum of the crossing signs in the part of the diagram representing the components. These expectation values are the *generalized regular isotopy invariants* of framed links. These can be exactly evaluated using the method developed in ref. [4]. The method makes use of the following two inputs:

1. Chern-Simons functional integral (containing Wilson lines) on a three-manifold with n-punctures on its boundary corresponds to a state in the space of n-correlator conformal blocks in the corresponding Wess-Zumino conformal field theory on that boundary [2].

2. Knots and Links can be obtained by closure of braids (Alexander theorem) or equivalently platting of braids (Birman theorem) [13].

Consider a manifold S^3 from which two non-intersecting three-balls are removed. This manifold has two boundaries, each an S^2. We place $2n$ Wilson line-integrals over lines connecting these two boundaries through a weaving pattern $\mathbf{B}$ as shown in the Figure (a) below. This is a $2n-$braid placed in this manifold. The strands are specified on the upper boundary by giving $2n$ assignments $(\hat{R}_1^*, \hat{R}_1, \hat{R}_2^*, \hat{R}_2, \ldots, \hat{R}_n^*, \hat{R}_n)$. Here $\hat{R} = (R, \epsilon)$ denotes representation R and orientation ϵ ($\epsilon = \pm 1$ for a strand going into or away from the boundary) and conjugate assignment $\hat{R}^* = (\bar{R}, -\epsilon)$ indicates reversal of the orientation. Similar specifications are done with respect to the lower boundary by the representation assignments $(\hat{\ell}_1, \hat{\ell}_1^*,\ \hat{\ell}_2, \hat{\ell}_2^*, \ldots.\hat{\ell}_n, \hat{\ell}_n^*)$. Then the assignments $\{\hat{\ell}_i\}$ are just a permutation of $\{\hat{R}_i^*\}$. Chern-Simons functional integral over this manifold is a state in the tensor product of the Hilbert spaces associated with the two boundaries, $\mathcal{H}_1 \otimes \mathcal{H}_2$. This state can be expanded in terms of some convenient basis[4, 6]. These bases are given by the conformal blocks for $2n$-point correlators of the associated $\mathcal{G}_k$ Wess-Zumino conformal field theory on each of the S^2 boundaries.



An arbitrary braid can be generated by a sequence of elementary braidings. The eigenvalues of these elementary braids are given by conformal field theory. The braiding eigenvalues depend on the framing. In vertical framing, the eigenvalues for a right-handed half-twist between two parallely oriented strands carrying representation $R,\ R'$ ($\epsilon\epsilon' = 1$) and for anti-parallely oriented strands ($\epsilon\epsilon' = -1$), as shown in the figure below, are respectively [4] :

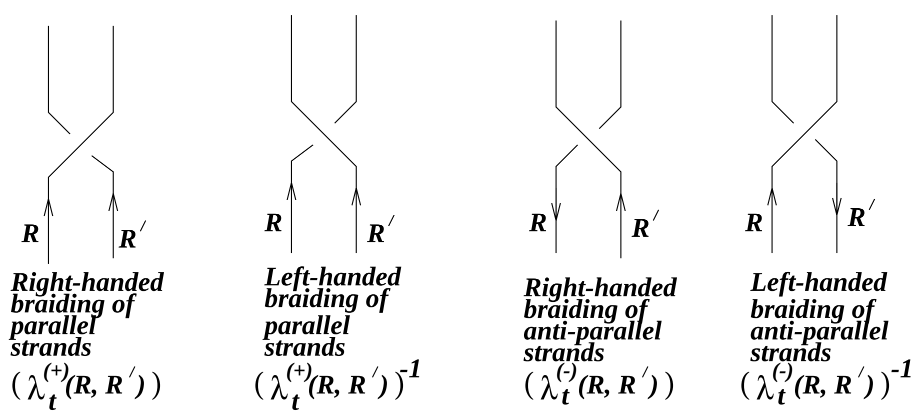
$$\lambda_t^{(+)}(R, R') = (-)^{\epsilon_R + \epsilon_{R'} - \epsilon_t}\ q^{-(C_R + C_{R'})/2 + C_t/2} \tag{5}$$

$$\lambda_t^{(-)}(R, R') = (-)^{|\epsilon_R - \epsilon_{R'}| - \epsilon_t}\ q^{(C_R + C_{R'})/2 - C_t/2}\ , \tag{6}$$

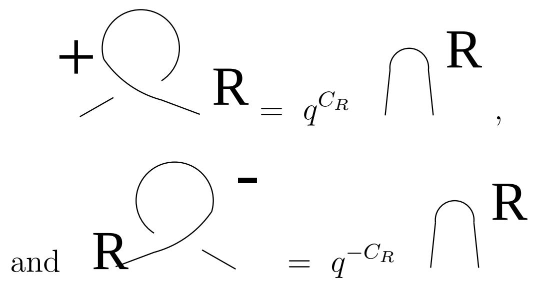
where t takes values allowed in the product of representations of R and R' given by the fusion rules of $\mathcal{G}_k$ Wess-Zumino conformal field theory, q-independent phases $(-)^{\epsilon_R+\epsilon_{R'}-\epsilon_t} = \pm 1$, $(-)^{|\epsilon_R-\epsilon_{R'}|-\epsilon_t} = \pm 1$, and $C_R, C_{R'}$ are the quadratic Casimir of representations R, R' respectively. The variable q in the above equation is related to the coupling constant k in Chern-Simons theory as

$$q = exp(\frac{2\pi i}{k + C_v})\ , \tag{7}$$

where C_v is quadratic Casimir of adjoint representation.



As mentioned earlier, the link invariants in vertical framing are only regular isotopy invariants. That is, these invariants do not remain unchanged when a writhe is smoothed out, but instead pick up a phase:



where $\pm$ indicate the right-handedness and left-handedness of the writhe.

Writing the weaving pattern $\mathbf{B}$ in Figure (a) above in terms of elementary braids, the Chern-Simons functional integral over this manifold is given by a matrix $\mathbf{B}(\{R_i\},\ \{\ell_i\})$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$. To plat this braid, we consider two balls with Wilson lines as shown in Figures (b) and (c) above. We glue these respectively from above and below onto the manifold of Figure (a). This yields a link in S^3.

The Chern-Simons functional integral over the ball (c) is given by a vector in the Hilbert space associated with its S^2 boundary. This vector $|\psi(\{\ell_i\})\rangle$ can again be written in terms of a convenient basis of this Hilbert space. Similarly, the functional integral over the ball of Figure (b) above is a dual vector $\langle\psi(\{R_i\})|$ in the associated dual Hilbert space. Gluing these two balls on to each other (along their oppositely oriented boundaries) gives n disjoint unknots carrying representations $R_1,\ R_2,\ ....\ R_n$ in S^3. Their invariant is simply the product of invariants for n individual unknots, due to the factorization property of invariants for disjoint knots. Thus the inner product of vectors representing the functional integrals over manifolds (b) and (c) is given by

$$\langle\psi(\{R_i\})|\psi(\{R_i\})\rangle \;=\; \prod_{i\,=\,1}^{n} V_{R_i}[U]\ , \tag{8}$$

where the invariant for unknot carrying representation R_i is given by $V_{R_i}[U] = dim_q\ R_i$ which is the quantum dimension for representation R_i defined in terms of highest weight Λ_{R_i}, Weyl vector ρ and positive roots α_+ [14]:

$$\dim_q\ R_i = \prod_{\alpha\in\Delta_+} \frac{[(\Lambda_{R_i},\alpha_+) + (\rho,\alpha_+)]}{[(\rho,\alpha_+)]} = \frac{S_{0\Lambda_{R_i}}}{S_{00}}\ . \tag{9}$$

Here the square brackets denote $q-$number defined as:

$$[x] \; = \; \frac{(q^{\frac{x}{2}} - q^{-\frac{x}{2}})}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \; , \tag{10}$$

with q as given in eqn. (7). Matrix S represents the generator of modular transformation $\tau \to -1/\tau$ on the characters of associated $\mathcal{G}_k$ Wess-Zumino conformal field theory. Its form for a group $\mathcal{G}$ of rank r and dimension d is given by [15, 16, 17, 18]:

$$S_{\Lambda_{R_1}\Lambda_{R_2}} = (-i)^{\frac{d-r}{2}} |\frac{L_\omega}{L}|^{-\frac{1}{2}} (k + C_v)^{-\frac{1}{2}} \sum_{\omega \in W} \epsilon(\omega) \exp\left(\frac{-2\pi i}{k + C_v}(\omega(\Lambda_{R_1} + \rho), \Lambda_{R_2} + \rho)\right) \; , \tag{11}$$

where W denotes the Weyl group and its elements ω are words constructed using the generator s_{α_i} – that is, $\omega = \prod_i s_{\alpha_i}$ and $\epsilon(\omega) = (-1)^{\ell(\omega)}$ with $\ell(\omega)$ as length of the word. The action of the Weyl generator s_α on a weight Λ_R is

$$s_\alpha(\Lambda_R) = \Lambda_R - 2\alpha \frac{(\Lambda_R, \alpha)}{(\alpha, \alpha)} \; , \tag{12}$$

and $|L_\omega/L|$ is the ratio of weight and co-root lattices (equal to the determinant of the cartan matrix for simply laced algebras).

Having determined the state corresponding to functional integrals over three-manifolds as drawn in Figs. (b), (c) above, we shall now obtain the invariant for a link obtained by gluing the two balls (b) and (c) on to the manifold of Figure (a). The link invariant is equal to the matrix element of matrix $\mathbf{B}$ between these two vectors. This can be calculated by generalizing the method of ref.([4, 6]) for arbitrary semi-simple groups through following proposition:

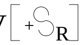
Proposition 1: *Expectation value of a Wilson operator for an arbitrary n component framed link $[L, \mathbf{f}]$ with a plat representation in terms of a braid $\mathbf{B}(\{R_i\}, \{\ell_i\})$ generated as a word in terms of the braid generators is given by*

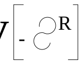
$$V[L; \mathbf{f}; R_1, R_2, \ldots R_n] \; = \; \langle \psi(\{R_i\}) | \; \mathbf{B}(\{R_i\}, \{\ell_i\}) \; |\psi(\{\ell_i\})\rangle \tag{13}$$

Thus these invariants for any arbitrary framed link can be evaluated.

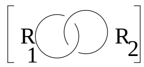
Examples

a) Unknot with framing number +1 or -1 is related to the unknot in zero framing as:

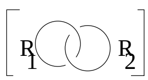
V  $= \quad q^{C_R}$ V  $= \; q^{C_R} \; dim_q R \qquad (14)$

and V  $= \quad q^{-C_R}$ V  $= \; q^{-C_R} \; dim_q R \; . \qquad (15)$

b) The invariant for a Hopf link carrying representation R_1 and R_2 on the component knots in vertical framing can be obtained in two equivalent ways using the braiding and inverse braiding (parallel strands):

V 

$$= \sum_\ell N^\ell_{R_1 R_2} dim_q \; \ell \; \left(\lambda^{(+)}_\ell(R_1, \; R_2)\right)^2 = q^{-C_{R_1}-C_{R_2}} \sum_\ell N^\ell_{R_1 R_2} \; dim_q \; \ell \;\; q^{C_\ell}, \tag{16}$$

V 

$$= \sum_\ell N^\ell_{R_1 R_2} dim_q \; \ell \; \left(\lambda^{(+)}_\ell(R_1, \; R_2)\right)^{-2} = q^{C_{R_1}+C_{R_2}} \sum_\ell N^\ell_{R_1 R_2} dim_q \; \ell \;\; q^{-C_\ell}, \tag{17}$$

where the summation over ℓ is over all the allowed representations in $\mathcal{G}_k$ conformal field theory and coefficients $N^\ell_{R_1 R_2}$ are given by the fusion rules of the conformal theory:

$$\begin{aligned} N^\ell_{R_1 R_2} &= 1 \qquad \text{if } \ell \in \mathrm{R}_1 \otimes \mathrm{R}_2 \;\; (\text{ fusion rules of } \mathcal{G}_\mathrm{k} \text{ conformal field theory}) \\ &= 0 \qquad \text{otherwise} \; . \end{aligned} \tag{18}$$

There is an explicit form for the fusion matrix in terms of elements of modular matrix S [19, 18, 17]:

$$N^\ell_{R_1 R_2} = \sum_m S_{\Lambda_{R_2}\Lambda_m} \left(\frac{S_{\Lambda_{R_1}\Lambda_m}}{S_{0\Lambda_m}}\right) S^*_{\Lambda_m \Lambda_\ell} \; , \tag{19}$$

where $\left(\frac{S_{\Lambda_{R_1}\Lambda_m}}{S_{0\Lambda_m}}\right)$ can be shown to be the eigenvalues of fusion matrix $(N_{R_1})^\ell_{R_2} \equiv N^\ell_{R_1 R_2}$. We can show the topological equivalence of Hopf links (16, 17) by exploiting the properties of generators S (11) and T $(\tau \to \tau + 1)$ representing modular transformations on the characters of associated $\mathcal{G}_k$ Wess-Zumino conformal field theory:

$$S^2 = C \; , \tag{20}$$

$$(ST)^3 = 1 \; , \tag{21}$$

where $C_{\Lambda\Lambda'} = \delta_{\Lambda'\bar{\Lambda}}$ is the charge conjugation matrix and $S^* = S^\dagger = S^{-1}$, $S^* = CS = SC$; $S^*_{\Lambda\Lambda'} = S_{\bar{\Lambda}\Lambda'} = S_{\Lambda\bar{\Lambda}'}$. Modular generator T has a diagonal form[15, 18]:

$$T_{\Lambda_{R_1}\Lambda_{R_2}} = \exp(\frac{-i\pi c}{12})\ q^{C_{\Lambda_{R_1}}}\ \delta_{\Lambda_{R_1}\Lambda_{R_2}}\ , \tag{22}$$

where central charge of the conformal field theory is $c = \frac{kd}{k+C_v}$ with d denoting dimension of the group $\mathcal{G}$. Eqn. (21) can be rewritten as

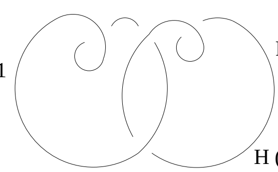
$$\sum_{\Lambda_\ell} S_{\Lambda_m\Lambda_\ell}\ q^{C_{\Lambda_\ell}}\ S_{\Lambda_\ell\Lambda_t} = \alpha\ q^{-C_{\Lambda_m}-C_{\Lambda_t}}\ S^*_{\Lambda_m\Lambda_t}, \tag{23}$$

where $\alpha = exp(\frac{i\pi c}{4})$. Using eqns. (9, 19, 23), we have:

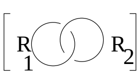
$$q^{-C_{R_1}-C_{R_2}} \sum_\ell\ dim_q\ \ell\ \ q^{C_\ell}\ \ =\ \ q^{C_{R_1}+C_{R_2}} \sum_\ell\ dim_q\ \ell\ \ q^{-C_\ell}\ \ =\ \ \frac{S_{\Lambda_{R_1}\Lambda_{R_2}}}{S_{00}}\ . \tag{24}$$

Here summation over ℓ runs over only those irreducible representations in the product $R_1 \otimes R_2$ which are allowed by fusion rules (eqn. 18). This confirms the equality of two expressions in eqns. (16) and (17) for the invariant for Hopf link.

(c) Next consider the Hopf link $H(R_1,\ R_2)$ with framing $+1$ for each of its component knots as drawn below. The framing is represented by a right-handed writhe in each of the knots.



The invariant for this link is given by

$$V[H(R_1,\ R_2)]\ =\ q^{C_{R_1}+C_{R_2}}\ \ \mathbf{V}\ \Big[$$


$$\Big]\ =\ q^{C_{R_1}+C_{R_2}}\ \ \frac{S_{\Lambda_{R_1}\Lambda_{R_2}}}{S_{00}}\ , \tag{25}$$

where the first factor $q^{C_{R_1}+C_{R_2}}$ comes from two writhes.

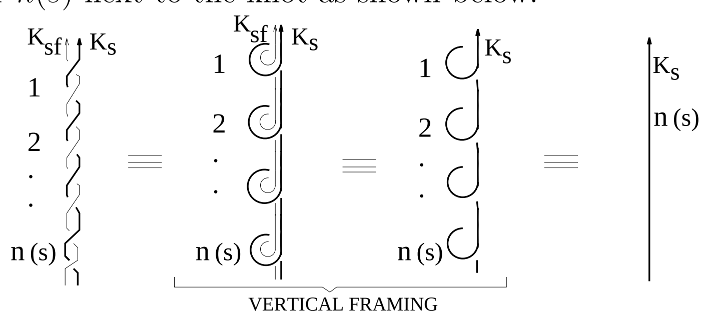
Next we shall present a discussion of how such invariants for framed links in S^3 based on any arbitrary group can be used to construct a manifold invariant for arbitrary three-manifolds. This will generalize a similar construction done earlier for $SU(2)$ group [6, 8].

3 Three-manifold invariants

We shall first recapitulate the mathematical details of how three-manifolds are constructed by surgery on framed unoriented links. This will subsequently be used to derive an algebraic formula in terms of the link invariants characterizing three-manifolds so constructed. Starting step in this discussion is a theorem due to Lickorish and Wallace [20, 21]:

Fundamental theorem of Lickorish and Wallace: *Every closed, orientable, connected three-manifold, M^3 can be obtained by surgery on an unoriented framed knot or link $[L,\ f]$ in S^3.*

As pointed out earlier framing f of a link L is defined by associating with every component knot K_s of the link an accompanying closed curve K_{sf} parallel to the knot and winding $n(s)$ times in the right-handed direction. That is the linking number $lk(K_s, K_{sf})$ of the component knot and its frame is $n(s)$. In so called vertical framing where the frame is thought to be just vertically above the two dimensional projection of the knot as shown below, we may indicate this by putting $n(s)$ writhes in the knot or even by just simply writing the integer $n(s)$ next to the knot as shown below:

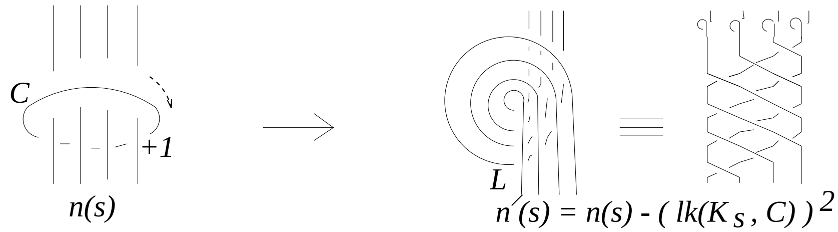


Next the surgery on a framed link $[L, f]$ made of component knots $K_1, K_2, ... K_r$ with framing $f = (n(1), n(2), ... n(r))$ in S^3 is performed in the following manner. Remove a small open solid torus neighbourhood N_s of each component knot K_s, disjoint from all other such open tubular neighbourhoods associated with other component knots. In the manifold left behind $S^3 - (N_1 \cup N_2 \cup\ .... N_r)$, there are r toral boundaries. On each such boundary, consider a simple closed curve (the frame) going $n(s)$ times along the meridian and once along the longitude of associated knot K_s. Now do a modular transformation on such a toral boundary such that the framing curve bounds a disc. Glue back the solid tori into the gaps. This yields a new manifold M^3. The theorem of Lickorish and Wallace assures us that every closed, orientable, connected three-manifold can be constructed in this way.

This construction of three-manifolds is not unique: surgery on more than one framed link can yield homeomorphic manifolds. But the rules of equivalence of framed links in S^3 which yield the same three-manifold on surgery are known. These rules are known as Kirby moves.

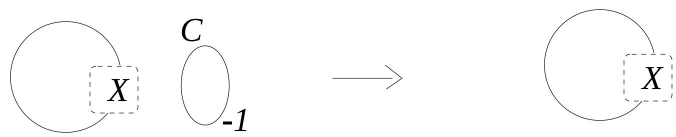
Kirby calculus on framed links in S^3: Following two elementary moves (and their inverses) generate Kirby calculus[22]:

Move I. For a number of unlinked strands belonging to the component knots K_s with framing $n(s)$ going through an unknotted circle C with framing $+1$, the circle can be removed after making a complete clockwise (left-handed) twist from below in the disc enclosed by circle C:



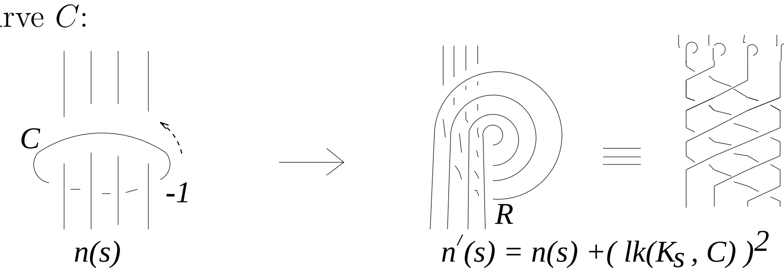
In the process, in addition to introducing new crossings, the framing of the various resultant component knots, K'_s to which the affected strands belong, change from $n(s)$ to $n'(s) \;=\; n(s) - (lk(K_s, C))^2$.

Move II. Drop a disjoint unknotted circle C with framing -1 without any change in rest of the link:



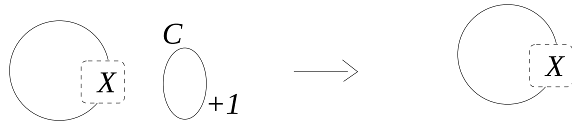
Two Kirby moves (I) and (II) and their inverses generate the conjugate moves[9]:

Move $\bar{I}$. Here a circle C with framing -1 enclosing a number strands can be removed after making a complete anti-clockwise (right-handed) twist from below in the disc bounded by curve C:



Again, this changes the framing of the resultant knots K'_s to which the enclosed strands belong from $n(s)$ to $n'(s) \ = \ n(s) + (lk(K_s, C))^2$.

Move $\bar{I}\bar{I}$. A disjoint unknotted circle C with framing $+1$ can be dropped without affecting rest of the link:



Thus Lickorish-Wallace theorem and equivalence of surgery under Kirby moves reduces the theory of closed, orientable, connected three-manifolds to the theory of framed unoriented links via a one-to-one correspondence:

$$\begin{pmatrix} Framed\ unoriented\ links\ in\ S^3\ modulo \\ equivalence\ under\ Kirby\ moves \end{pmatrix} \leftrightarrow \begin{pmatrix} Closed,\ orientable,\ connected\ three- \\ manifolds\ modulo\ homeomorphisms \end{pmatrix}$$

This consequently allows us to characterize three-manifolds by the invariants of associated unoriented framed knots and links obtained from the Chern-Simons theory in S^3. This can be done by constructing an appropriate combination of the invariants of framed links which is unchanged under Kirby moves and which does not see orientations of the framed link:

$$\begin{pmatrix} Combination\ of\ framed\ \ link\ invariants \\ which\ do\ not\ change\ under\ Kirby\ moves \end{pmatrix} = \begin{pmatrix} Invariants\ of\ associated \\ three-manifold \end{pmatrix}$$

Using the framed link invariants presented in the previous section, we shall now construct such a three-manifold invariant which is preserved under Kirby moves. The immediate step in this direction is to construct a combination of these link invariants which would be unchanged under Kirby move I:

In order to achieve this, we have to first solve the following equation for μ_{R_2} and β relating the invariants $V[H(R_1, R_2)]$ and $V[U(R_1)]$ for these two links respectively:

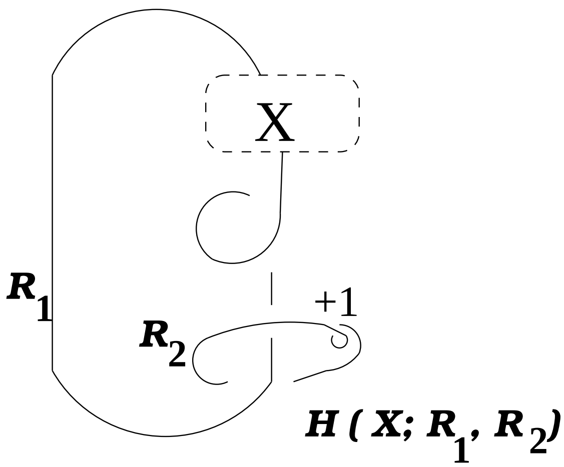
$$\sum_{R_2} \ \mu_{R_2} \ V[H(R_1, R_2)] \ = \ \beta \ V[U(R_1)] \ , \tag{26}$$

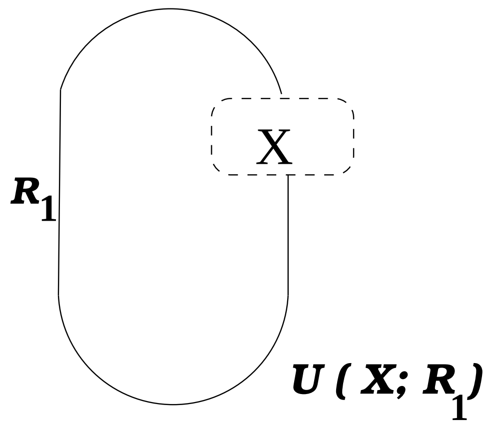
where summation R_2 is over all the representations (highest weight Λ_{R_2}) with projection along the longest root θ as $(\Lambda_{R_2}, \theta) \le k$. These are all the integrable representations

of $\mathcal{G}_k$ conformal field theory. Rewriting the framed link invariants in terms of modular transformation matrix S (25, 9) and comparing with the identity (23), we deduce the following solution:

$$\mu_{R_2} \;=\; S_{0\Lambda_{R_2}} \;, \qquad \beta = \alpha \equiv e^{\pi i c/4} \;. \tag{27}$$

Next we will consider the following two links $H(X;\ R_1,\ R_2)$ and $U(X;\ R_1)$:





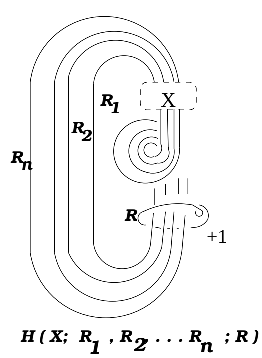
where X as an arbitrary entanglement inside the box. The link $H(X;\ R_1,\ R_2)$ is connected sum of the link $U(X;\ R_1)$ and a framed Hopf link $H(R_1,\ R_2)$. Factorization properies of invariants of such a connected sum of links yields:

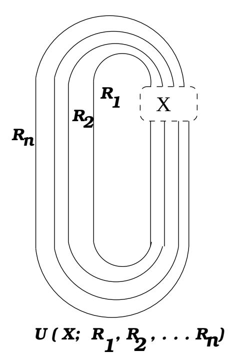
$$dim_q\ R_1\ \ V[H(X;\ R_1,\ R_2)] \;=\; V[U(X;\ R_1)]\ \ V[H(R_1,\ R_2)] \;. \tag{28}$$

Using eqn. (26), this further implies (note $dim_q\ R_1$ is the invariant $V[U(R_1)]$ for unknot in representation R_1 and with zero framing):

$$\sum_{R_2} \mu_{R_2}\ V[H(X;\ R_1,\ R_2)] \;=\; \alpha\ V[U(X;\ R_1)] \;. \tag{29}$$

We can generalize this relation for the following links $H(X;\ R_1,\ R_2,\ ....\ R_n;\ R)$ and $U(X;\ R_1,\ R_2,\ ....R_n)$,





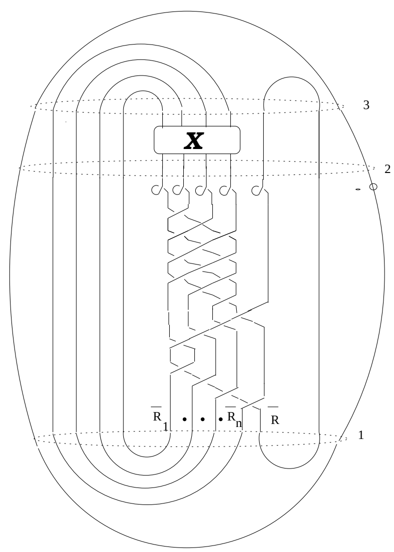
as

Proposition 2: *The invariants for these two links are related as:*

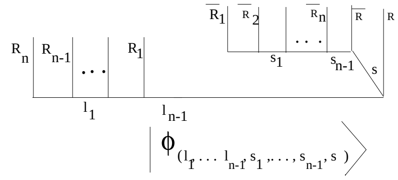
$$\sum_R \mu_R \, V[H(X;\ R_1,\ R_2, ...\ R_n;\ R)] \ = \ \alpha \, V[U(X;\ R_1,\ R_2, ...\ R_n)] \ . \tag{30}$$

Thus this proposition provides for equivalence of the two links under Kirby move I up to a phase factor α on the right-hand side. Lets us now outline the proof of this proposition.

Proof: In the following figure, we have redrawn the link $H(X; R_1, R_2, \ldots R_n;\ R)$ in S^3 by gluing four three-manifold: two three-balls (each with S^2 boundary) and two three-manifolds with two S^2 boundaries each. The various boundaries have been glued together along the dotted lines as indicated. This allows us to use the Proposition 1 above to evaluate the invariant for this link.



Instead of plat diagram described in Section 2, we have the closure of a braid here. The states corresponding to Chern-Simons functional integral over two three-balls with S^2 boundaries '1' and '3' will be represented by the vector $|\psi(\{R_i\}, R)\rangle$ (where $i \in [1, n]$) and its dual. The matrix element corresponding to the braiding inside the three-manifold with two S^2 boundaries '1' and '2' can be explicitly computed in a convenient basis. We shall work with a basis represented by the following conformal block of associated Wess-Zumino theory:



where $l_1 = R_n \otimes R_{n-1}, \ldots l_{n-1} = l_{n-2} \otimes R_1$ and $s_1 = R_1 \otimes R_2, \ldots s_{n-1} = s_{n-2} \otimes R_n, s = l_{n-1} \otimes R$. In this basis the matrix corresponding to the three-manifold with boundaries marked '1' and '2' turns out to be:

$$\nu_1 = \sum_{l_1,\ldots l_{n-1},s_1,\ldots s_{n-1},s} q^{C_s} \quad |\phi^{(2)}_{(l_1,\ldots l_{n-1},s_1,\ldots s_{n-1},s)}\rangle\langle\phi^{(1)}_{(l_1,\ldots l_{n-1},s_1,\ldots s_{n-1},s)}| \tag{31}$$

where the superscripts (1) and (2) inside the basis states refer to two S^2 boundaries containing the three-manifold. This result involves properties of the braiding and duality matrices which we present for $n = 2$ in the Appendix. These can readily be generalized to arbitrary n.

The matrix $\nu_2(X)$ representing the other three-manifold containing entanglement X between two S^2 boundaries indicated by dotted lines '2' and '3' in the figure above can similarly be evaluated. In addition, we also need to write down the states $|\psi^{(1)}\rangle$ and $\langle\psi^{(3)}|$ corresponding to the two three-balls with boundaries indicated by dotted lines '1' and '3'. All these in the above basis can be written as :

$$\nu_2(X) = \sum_{\{l_i\},s,\{s_i\},\{s'_i\}} X(\{s_i\},\ \{s'_i\}) \quad |\phi^{(3)}_{(l_1,\ldots l_{n-1},s_1,\ldots s_{n-1},s)}\rangle\langle\phi^{(2)}_{(l_1,\ldots l_{n-1},s'_1,\ldots s'_{n-1},s)}| \tag{32}$$

$$|\psi^{(1)}\rangle = \sum_{l_1,\ldots l_{n-1},,s} \sqrt{dim_q\ s} \quad |\phi^{(1)}_{(l_1,\ldots l_{n-1},l_1,\ldots l_{n-1},s)}\rangle \tag{33}$$

$$\langle\psi^{(3)}| = \sum_{l_1,\ldots l_{n-1},s} \sqrt{dim_q\ s} \quad \langle\phi^{(3)}_{(l_1,\ldots l_{n-1},l_1,\ldots l_{n-1},s)}| \tag{34}$$

Gluing these four three-manifolds along the oppositely oriented S^2 boundaries, we get the link $H(X; R_1, R_2, \ldots R_n; R)$ whose invariant can now be written as:

$$V[H(X; R_1, R_2 \ldots R_n; R)] = \sum_{\{l_i\}} X(\{l_i\},\ \{l_i\}) \left[\sum_{s\in l_{n-1}\otimes R} (dim_q\ s) \quad q^{C_s} \right] . \tag{35}$$

Clearly the term in square bracket is the Hopf link invariant $H(l_{n-1}, R)$ of eqn. (25). Similarly, we can compute the link invariant for $U(X; R_1, R_2 \ldots R_n)$ as

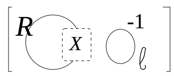
$$V[U(X; R_1, R_2 \ldots R_n)] = \sum_{\{l_i\}} (dim_q\ l_{n-1}) \quad X(\{l_i\},\ \{l_i\}) . \tag{36}$$

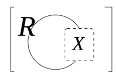
Now using eqn.(26), it is easy to prove Proposition 2:

$$\sum_R \mu_R\ V[H(X; R_1, R_2 \ldots R_n; R)] = \sum_{\{l_i\}} X(\{l_i\},\ \{l_i\}) \sum_R \mu_R\ V[H(l_{n-1}, R)] \tag{37}$$

$$= \alpha\ V[U(X; R_1, R_2 \ldots R_n)] . \tag{38}$$

This completes our discussion of Kirby move I. For Kirby move II, we note that for a link containing a disjoint unknot with framing -1, we have:

$$\sum_{\ell} \mu_\ell \, \mathbf{V}$$


$$= \alpha^* \, \mathbf{V}$$


(39)

This follows readily due to the exact factorizations of invariants of disjoint links into those of individual links and use of the identity: $\sum_\ell S_{0\ell} \, q^{-C_\ell} \, S_{\ell 0} = e^{-\pi i c/4} \, S_{00}$.

Clearly the Eqns. (30) and (39) respectively correspond to the two generators of Kirby calculus. Presence of phases α in these equations reflects change of three-framing of the associated manifold under Kirby moves. Three-manifolds constructed by surgery on links equivalent under Kirby moves though topologically same, yet may differ in terms of their three-framing. The dependence on three-framing (α factors in above equations representing Kirby moves) can be rotated away by use of a nice property of the linking matrix under Kirby moves. For a framed link $[L, \ f]$ whose component knots $K_1, \ K_2, \ .... \ K_r$ have framings (self-linking numbers) as $n_1, \ n_2, \ .... \ n_r$ respectively, the linking matrix is defined as

$$W[L, \ f] \ = \ \begin{pmatrix} n_1 & lk(K_1, K_2) & lk(K_1, K_3) & ..... & lk(K_1, K_r) \\ lk(K_2, K_1) & n_2 & lk(K_2, K_3) & ..... & lk(K_2, K_r) \\ .. & .. & n_3 & ..... & .. \\ .. & .. & .. & ..... & .. \\ lk(K_r, K_1) & .. & .. & ..... & n_r \end{pmatrix}$$

where $lk(K_i, K_j)$ is the linking number of knots K_i and K_j. The signature of linking matrix is given by

$$\sigma[L, \ f] \ = \ (\text{no. of } +\text{ve eigenvalues of } W) - (\text{no. of } -\text{ve eigenvalues of } W)$$

Then this signature for the framed link $[L, \ f]$ and those for the links $[L', \ f']$ obtained by transformation under two elementary generators of Kirby calculus are related in a simple fashion:

$$\begin{aligned} \text{Kirby move I}: \qquad \sigma[L, \ f] \ &= \ \sigma \ [L', \ f'] + 1 \ ; \\ \text{Kirby move II}: \qquad \sigma[L, \ f] \ &= \ \sigma \ [L', \ f'] - 1 \ . \end{aligned} \tag{40}$$

Notice, though the sign of linking numbers $lk(K_i, K_j)$ for distinct knots does depend on the relative orientations of knots K_i and K_j, the signature of linking matrix does not depend on the relative orientations of component knots.

Now collecting the properties of framed link (30, 39) and the signature of linking matrix under the Kirby moves (40), we may state our main result:

Proposition 3: *For a framed link $[L,\ f]$ with component knots, $K_1,\ K_2,\ ....\ K_r$ and their framings respectively as $n_1,\ n_2,\ ....\ n_r$, the quantity*

$$\hat{F}[L,\ f] \ = \ \alpha^{-\sigma[L,\ f]} \sum_{\{R_i\}} \ \mu_{R_1}\ \mu_{R_2}\ ....\ \mu_{R_r}\ V[L;\ n_1,\ n_2,\ ...\ n_r;\ R_1,\ R_2,\ ....\ R_r] \quad (41)$$

constructed from invariants V (in vertical framing) of the framed link, is an invariant of the associated three-manifold obtained by surgery on that link.

Notice individual link invariants $V[L; n_1, n_2, \ldots n_r; R_1, \ldots R_r]$ do in general depend on the relative orientations of component knots. Reversal of orientation on a particular knot changes the group representation living on it to its conjugate. Since all representations are summed for each component knot, the resultant combination (41) is an invariant of unoriented link.

The combination $\hat{F}[L,\ f]$ of link invariants so constructed is exactly unchanged under Kirby calculus. Because of the factor depending on signature of linking matrix in front, there are no extra factors of α generated by Kirby moves.

This generalizes a similar proposition obtained earlier[6, 8] for $SU(2)$ theory to any arbitrary semi-simple gauge group.

Explicit examples: Now let us give the value of this invariant for some simple three-manifolds.

1) The surgery descriptions of manifolds S^3, $S^2 \times S^1$, RP^3 and Lens spaces of the type $\mathcal{L}(p, \pm 1)$ are given by an unknot with framing $+1$, 0 , $+2$ and $\pm p$ respectively. As indicated above the knot invariant for an unknot with zero framing carrying representation R is $dim_q\ R = S_{\Lambda_R 0}/S_{00}$. Thus the invariant for S^3 is:

$$\hat{F}[S^3] \ = \ \alpha^{-1} \sum_{\Lambda_R} \ S_{0\Lambda_R}\ q^{C_{\Lambda_R}}\ \frac{S_{\Lambda_R 0}}{S_{00}}\ ,$$

where the factor $q^{C_{\Lambda_R}}$ comes from framing $+1$ (one right-handed writhe). Use of identity (23), $\sum_{\Lambda_R}\ S_{0\Lambda_R}\ q^{C_{\Lambda_R}}\ S_{\Lambda_R 0} = \alpha\ S_{00}$ immediately yields the invariant simply to be:

$$\hat{F}[S^3] \ = \ 1 \quad (42)$$

Next for three-manifold $S^2 \times S^1$, we have

$$\hat{F}[S^2 \times S^1] \ = \ \sum_{\Lambda_R} \ S_{\Lambda_R 0}\ \frac{S_{\Lambda_R 0}}{S_{00}} \ = \ \frac{1}{S_{00}}\ , \quad (43)$$

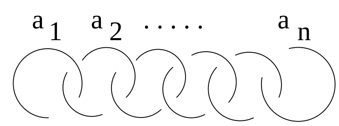
where orthogonality of the S-matrix has been used. Finally, for RP^3 and more generally Lens spaces $\mathcal{L}(p, \pm 1)$, we have

$$\hat{F}[RP^3] \;=\; \alpha^{-1} \sum_R \frac{S_{0\Lambda_R}\; q^{2C_{\Lambda_R}}\; S_{\Lambda_R 0}}{S_{00}}\;, \tag{44}$$

$$\hat{F}[\mathcal{L}(p, \pm 1)] \;=\; \alpha^{\mp 1} \sum_R \frac{S_{0\Lambda_R}\; q^{\pm p C_{\Lambda_R}}\; S_{\Lambda_R 0}}{S_{00}}\;. \tag{45}$$

2) A more general example is the whole class of Lens spaces $\mathcal{L}(p,\ q)$; above manifolds are special cases of this class. These are obtained[21] by surgery on a framed link made of successively linked unknots with framing given by integers $a_1,\ a_2,\ \ldots\ldots\ a_n$:

$$[L, f] \quad =$$



where these framing integers provide a continued fraction representation for the ratio of two integers $p,\ q$:

$$\frac{p}{q} \;=\; a_n - \cfrac{1}{a_{n-1} - \cfrac{1}{\ldots\ldots\ a_3 - \cfrac{1}{a_2 - \frac{1}{a_1}}}}\;.$$

The invariant for these manifolds can readily be evaluated. The relevant link $[L,\ f]$ above is just a connected sum of framed $n-1$ Hopf links so that its link invariant is obtained by the factorization property of invariants for such a connected sum. Placing representations $R_1,\ R_2,\ \ldots\ldots\ R_n$ on the component knots, this link invariant is:

$$V[L,\ f;\ R_1, R_2, \ldots.\ R_n] \;=\; \frac{q^{\sum_{i=1}^n a_i C_{R_i}}\;\; \prod_{i=1}^{n-1} S_{\Lambda_{R_i}\Lambda_{R_{i+1}}}}{S_{00}\;\; \prod_{i=2}^{n-1}\;\; S_{\Lambda_{R_i}0}}, \tag{46}$$

where the factor $q^{\sum_{i=1}^n a_i C_{R_i}}$ is due to the framing $f = (a_1,\ a_2, \ldots.\ a_n)$ of knots. This finally yields a simple formula for the three-manifold invariant:

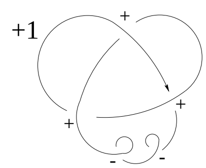
$$\hat{F}[\mathcal{L}(p,\ q)] \;=\; \alpha^{-\sigma[L,\ f]}\;\alpha^{(\sum a_i)/3}\;\; \frac{(S\ M^{(p,\ q)})_{00}}{S_{00}}\;, \tag{47}$$

where matrix $M^{(p,\ q)}$ is given in terms of the modular matrices S and T:

$$M^{(p,\ q)} \;=\; T^{a_n} S\ T^{a_{n-1}} S\ \ldots\ldots\ T^{a_2} S\ T^{a_1} S\;. \tag{48}$$

The corresponding expression for the partition function for $SU(2)$ Chern-Simons field theory in Lens spaces has also been obtained earlier in refs. [23].

3) Another example we take up is the Poincare manifold P^3 (also known as dodecahedral space or Dehn's homology sphere). It is given [21] by surgery on a right-handed trefoil knot with framing +1:



Notice, each right-handed crossing of the trefoil introduces +1 linking number between the knot and its vertical framing, and each of the two left-handed writhes contributes −1 so that the total frame number is +1. The knot invariant for a right-handed trefoil (in vertical framing with no extra writhes) carrying representation ℓ is:

$$V[T;\ \ell] = \sum_R N_{\ell\ell}^R dim_q\ R\ (-)^{6\epsilon_\ell - 3\epsilon_R}\ q^{-3C_\ell + \frac{3}{2}C_R}$$

Using this trefoil invariant and the Proposition 3, the three-manifold invariant for Poincare manifold turns out to be:

$$\hat{F}[P^3]\ =\ \alpha^{-1}\ \sum_{m,\ell,R} \frac{(-)^{6\epsilon_\ell - 3\epsilon_R} S_{0\Lambda_\ell}\ S_{0\Lambda_R}\ S_{\Lambda_\ell\Lambda_m}\ S_{\Lambda_\ell\Lambda_m}\ \ S^*_{\Lambda_R\Lambda_m}\ q^{-5C_\ell + \frac{3}{2}C_R}}{S_{00}\ S_{0\Lambda_m}}\ . \tag{49}$$

4) Similarly, surgery on a right-handed trefoil T with framing number −1 (that is, with four left-handed writhes in vertical framing) yields another homology three-sphere (with fundamental group presented by $\alpha, \beta : (\alpha\beta)^2 = \alpha^3 = \beta^7$). Its invariant is

$$\hat{F}[T, -1] = \alpha\ \sum_{m,\ell,R} \frac{(-)^{6\epsilon_\ell - 3\epsilon_R} S_{0\Lambda_\ell}\ S_{0\Lambda_R}\ S_{\Lambda_\ell\Lambda_m}\ S_{\Lambda_\ell\Lambda_m}\ S^*_{\Lambda_R\Lambda_m}\ q^{-7C_\ell + \frac{3}{2}C_R}}{S_{00}\ S_{0\Lambda_m}}\ . \tag{50}$$

5) The surgery on a right-handed trefoil with framing number +3 yields a coset manifold S^3/T^* where T^* is the binary tetrahedral group generated by two different $2\pi/3$ rotations α, β about two different vertices of a tetrahedron with $\alpha^3 = \beta^3 = (\alpha\beta)^2 = 1$. The three-manifold invariant for this coset manifold is

$$\hat{F}[S^3/T^*] = \alpha^{-1}\ \sum_{m,\ell,R} \frac{(-)^{6\epsilon_\ell - 3\epsilon_R} S_{0\Lambda_\ell}\ S_{0\Lambda_R}\ S_{\Lambda_\ell\Lambda_m}\ S_{\Lambda_\ell\Lambda_m}\ \ S^*_{\Lambda_R\Lambda_m}\ q^{-3C_\ell + \frac{3}{2}C_R}}{S_{00}\ S_{0\Lambda_m}}\ . \tag{51}$$

4 Conclusions

We have presented here a construction of a class of three-manifold invariants, one each for any arbitrary semi-simple gauge group. The construction exploits the one-to-one

correspondence between framed unoriented links in S^3 modulo equivalence under Kirby moves to closed orientable connected three-manifolds modulo homeomorphisms. Three-manifolds are characterized by an appropriate combination of invariants of the associated links. This combinations of link invariants is obtained from Chern-Simons theory in S^3 and is unchanged by Kirby moves. The construction is a direct generalization of that developed for an $SU(2)$ Chern-Simons theory earlier[6, 8]. The manifold invariant obtained from $SU(2)$ theory has been shown to be related to partition function of Chern-Simons theory on that manifold [11]. The generalized three-manifold invariants $\hat{F}(M)$ constructed here are also related to the partition function $Z(M)$ of Chern-Simons theory by an overall normalization:

$$\hat{F}(M) = \frac{Z(M)}{S_{00}} \ . \tag{52}$$

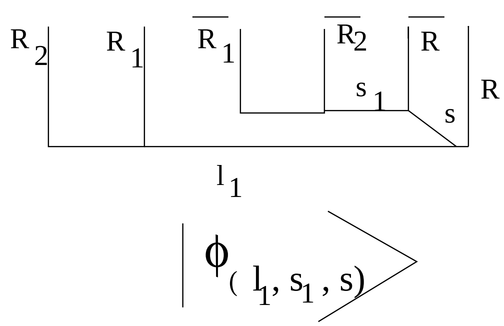
Thus, given a surgery presentation of a three-manifold, this provides a simple method of computing partition function of Chern-Simons field theory based on an arbitrary gauge group in that three-manifold.

Acknowledgements: P.R would like to thank TIFR for hospitality where part of this work got done. The work of P.R is supported by a CSIR Grant.

Appendix:

In this appendix, we shall use the properties of braiding and duality matrices to derive the result (31). for the case $n = 2$.

A convenient basis state for $n = 2$ is provided by the conformal block of the Wess-Zumino theory pictorially represented as:



The matrix ν_1 for $n = 2$ corresponds to a functional integral over a three-manifold with two S^2 boundaries, each with six punctures carrying representations R_2, R_1, R_1, R_2,

R and R.

Between these two boundaries we have braiding $b_4\ b_3^2\ b_4\ b_3^2$ and a $+1$ writhe on each of the three strands carrying representations $\bar{R}_1$, $\bar{R}_2$, $\bar{R}$. Using the explicit representation for braid generators in terms of their eigenvalues and duality matrices $a_{sp}\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$, we obtain ν_1 to be:

$$\begin{aligned} \nu_1 &= q^{C_{R_1}+C_{R_2}+C_R}\ b_4\ b_3^2\ b_4\ b_3^2\ \ |\phi^{(1)}_{l_1,s_1,s}\rangle\langle\phi^{(2)}_{l_1,s_1,s}| \\ &= q^{C_{R_1}+C_{R_2}+C_R}\ (\lambda_{s_1}(R_1,R_2))^2 \sum_{p_1 s_1',r,s_1''} a_{s_1 p}\begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ \bar{R} & s \end{bmatrix} \lambda_p(R_2,R) \\ &\quad a_{s_1' p}\begin{bmatrix} \bar{R}_1 & \bar{R} \\ \bar{R}_2 & s \end{bmatrix} (\lambda_{s_1'}(R_1,R))^2\ a_{s_1' r}\begin{bmatrix} \bar{R}_1 & \bar{R} \\ \bar{R}_2 & s \end{bmatrix} \lambda_r(R,R_2)\ a_{s_1'' r}\begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ \bar{R} & s \end{bmatrix} \\ &\quad |\phi^{(1)}_{l_1,s_1'',s}\rangle\langle\phi^{(2)}_{l_1,s_1,l}| \end{aligned} \tag{53}$$

Using the following property of the duality matrices,

$$\begin{aligned} \sum_l (-1)^{\epsilon_l}\ a_{pl}\begin{bmatrix} R_1 R_4 \\ R_3 R_2 \end{bmatrix}\ q^{\frac{C_l}{2}}\ a_{p'l}\begin{bmatrix} R_1 R_3 \\ R_4 R_2 \end{bmatrix} \ &= \ (-1)^{\epsilon_{R_1}+\epsilon_{R_2}+\epsilon_{R_3}+\epsilon_{R_4}-\epsilon_p-\epsilon_{p'}} \\ &\quad a_{pp'}\begin{bmatrix} R_3 R_2 \\ R_4 R_1 \end{bmatrix}\ q^{\frac{-C_p-C_{p'}+C_{R_1}+C_{R_2}+C_{R_3}+C_{R_4}}{2}} \end{aligned} \tag{54}$$

and the orthogonality relation, the above equation can be simplified to give

$$\nu_1 = \sum_{l_1,s_1,s} q^{C_s}\ |\phi^{(2)}_{l_1,s_1,s}\rangle\langle\phi^{(1)}_{l_1,s_1,s}| \tag{55}$$

Generalization of this result for arbitrary n is straight forward.

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